# A Study of Chaos in **Cellular Automata**

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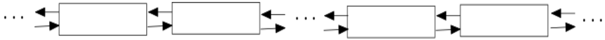
# Cellular Automata

- A discrete dynamical model consisting of infinitely many cells which are placed over $\mathbb{Z}$ and can evolve in discrete time and space.



- A cell can assume a state from a set $S = \{0, 1, \cdots, d-1\}$ and it moves to its next state depending on the present states of its left and right neighbors and itself.
- Therefore, we have the next state (NS) function $f : S^3 \to S$.
- The function is known as local transition function and it induces the global transition function $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ which can be defined as, for any $i \in \mathbb{Z}$, $F(x)_i = f(x_{i-1}, x_i, x_{i+1})$.

# Cellular Automata

- The next state function of the $i^{th}$ cell can be expressed in the tabular form

Table: The local transitions of rule 102

| PS<br>RMT | 111<br>(7) | 110<br>(6) | 101<br>(5) | 100<br>(4) | 011<br>(3) | 010<br>(2) | 001<br>(1) | 000<br>(0) | Rule |
|---|---|---|---|---|---|---|---|---|---|
| NS | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 102 |

**Definition**

*(Rule Min Term (RMT)): Let $f : S^3 \rightarrow S$ be the next state function of a CA. The tuple $\langle x_1, x_2, x_3 \rangle \in S^3$ is called a Rule Min Term or RMT, which is generally represented by its decimal equivalent $r = x_1.d^2 + x_2.d + x_3$.*

# Cellular Automata

## Definition

*(RMT Sequence): Let $x = (x_i)_{i \in \mathbb{Z}}$ be a configuration of a CA. The* **RMT sequence** *of $x$, denoted as $\tilde{x}$, is $(r_i)_{i \in \mathbb{Z}}$ where $r_i$ is the RMT $(x_{i-1}, x_i, x_{i+1})$.*

## Example

*Let $x = 1010$ be a configuration for a 4-cell 3-neighborhood CA. Then the RMT sequence corresponding to this configuration is $\tilde{x} = \langle 010(2), 101(5), 010(2), 101(5) \rangle$.*

# Cellular Automata

**Definition**

*(Equivalent RMT): A set of d RMTs* $r_1, r_2, \cdots, r_d$ *of a d-state CA rule are said to be equivalent to each other if* $r_1 d \equiv r_2 d \equiv \cdots \equiv r_d d \pmod{d^3}$.

**Definition**

*(Sibling RMT): A set of d RMTs* $s_1, s_2, \cdots, s_d$ *of a d-state CA rule are said to be equivalent to each other if* $\lfloor \frac{s_1}{d} \rfloor = \lfloor \frac{s_2}{d} \rfloor = \cdots = \lfloor \frac{s_d}{d} \rfloor \pmod{d^3}$.

For a 2-state CA $Equi_0 = \{000, 100\}$ and $Sibl_0 = \{000, 001\}$. In case of a 3-state CA, $Equi_0 = \{000, 100, 200\}$ and $Sibl_0 = \{000, 001, 002\}$.

# Cellular Automata

**Definition**

*(L-set): Let $r$ be the decimal equivalent of RMT $xyz$ where $x$, $y$, $z \in S$. Then, L-set($r$) $= \{$RMT $xy'z' | y' \neq y$ and $z' \neq z$, $y', z' \in S\}$.*

**Definition**

*(R-set): Let $r$ be the decimal equivalent of RMT $xyz$ where $x$, $y$, $z \in S$. Then, R-set($r$) $= \{$RMT $x'y'z | x' \neq x$ and $y' \neq y$, $x', y' \in S\}$.*

In case of a 3-state CA, the L-set for RMT 0 is L-set(0) $=$ {011, 012, 021, 022} and the R-set for RMT 0 is R-set(0) $=$ {110, 120, 210, 220}.

# Cellular Automata

## Definition

*An RMT xyz ($x, y, z \in S$) of a CA rule is called passive if $f(xyz) = y$. On the other hand, if $f(xyz) = y'$ ($y \neq y', y' \in S$), it is active RMT.*

For example, in ECA rule 102, the RMTs 0 (000), 2 (010), 4 (100) and 6 (110) are passive, whereas the rest RMTs are active.

## Definition

*A* **fixed-point attractor** *is a configuration in CA, for which the next configuration is itself. That means, if a CA reaches a fixed-point attractor, then it remains at that particular configuration forever.*

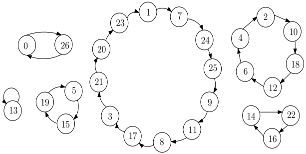
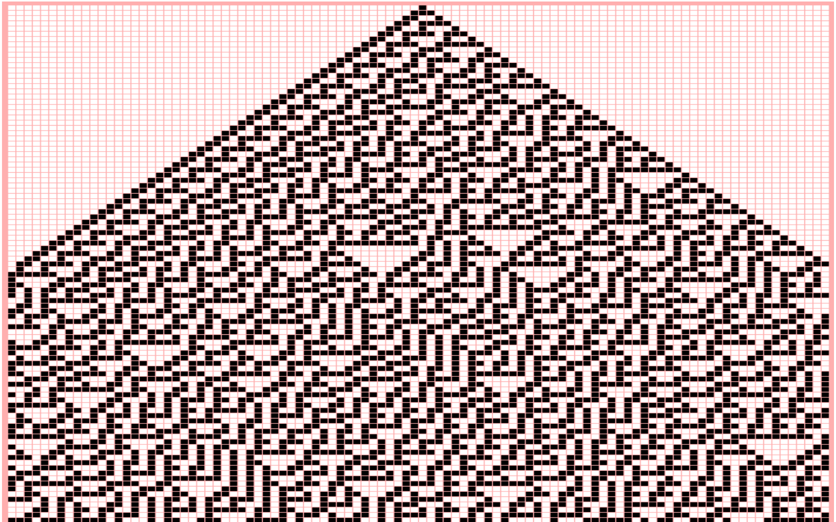



Figure: Showing Different attractors

# Cellular Automata and Chaos



# The source of unpredictability

- Add a bit of information to a cell and check whether it propagates to its neighbors. This is checked by two ways –
  - Information Propagation
  - Information Cooking

**Information Propagation** Assume that the state of a cell has been changed. We can now examine how this update affects its neighbors at the next time steps.
This can easily be understood by looking at the sibling RMTs and equivalent RMTs of the given rule.

# Information Propagation

- Consider the present state of cell $i \in \mathbb{Z}$ is $s_i$, which is updated to $s_i'$. If $f(s_{i-2}, s_{i-1}, s_i)$ and $f(s_{i-2}, s_{i-1}, s_i')$ are same, then it is concluded that the change of state of cell $i$ from $s_i$ to $s_i'$ has no effect on cell $i-1$.
- The RMTs $s_{i-2}s_{i-1}s_i$ and $s_{i-2}s_{i-1}s_i'$ are sibling to each other.
- To measure the possibility of getting affected of cell $i-1$ by cell $i$, we can use the following equations,

$$\delta_j^{(-1)}(r,s) = \begin{cases} 1 & \text{if } f_{i-1}(r) \neq f_{i-1}(s),\ r,s \in Sibl_j \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where $\delta_j^{(-1)} : Sibl_j \times Sibl_j \rightarrow \{0,1\}$ and $Sibl_j$ is a set of sibling RMTs.

# Information Propagation

$$A_j^{(-1)} = \frac{1}{d^2 - d} \sum_{\substack{r,s \in Sibl_j \\ r \neq s}} \delta_j^{(-1)}(r,s) \tag{2}$$

To find out the total possibility of information propagation to the left of cell $i$, we use the following parameter.

$$\Lambda_p = \frac{1}{d^2} \sum_{j=0}^{d^2-1} A_j^{(-1)} \tag{3}$$

We claim that $\Lambda_p$ is a measure of information propagation to the left neighbors of cell $i$.

## Example

*Consider the ECA rule 102. To calculate the value $\delta_0^{(-1)}$ we take the sibling RMT set $Sibl_0 = \{0, 1\}$. In that case, $\delta_0^{(-1)}(0,1) = 1$ as the next state values of both the RMTs are different. Similarly, we can calculate the values of $\delta_j^{(-1)}$ for all the sibling RMT sets. From these values, we get $A_0^{(-1)} = A_1^{(-1)} = A_2^{(-1)} = A_3^{(-1)} = 1$. Finally, the value of $\Lambda_p = 1$.*

# Information Propagation

- Using the same rationale, one can find another measure, say $\eta_p$
- $\eta_p$ indicates the possibility of information propagation to the right cell $i+1$ when state of cell $i$ is changed.
- Here we consider equivalent RMT sets for the right cell of cell $i$.

**Example**

*Consider the ECA rule 102. With the help of the above equations one can calculate the information flow in right direction ($\eta_p$). For the rule 102, the value of $\eta_p = 0$. This means the rule does not have any information flow to the right direction of the system.*

# Information Propagation

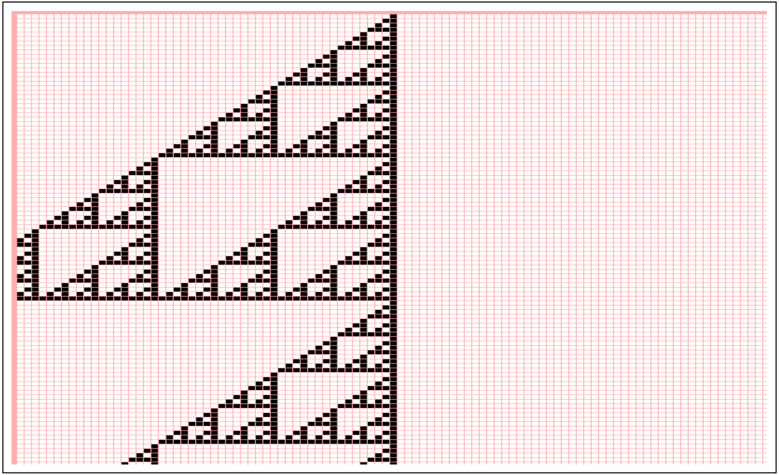


Figure: Space-time diagram of rule 102

# Information Cooking

- Now we examine how the update affects the cell itself
- When left neighbor or right neighbor is updated in a reaction to the update of cell $i$, the cell $i$ will further be updated in next time step in response to the updates of its neighbors. We call this fact as, *information cooking*.
- To guess the impact of the update on the cell itself from the rule, the RMTs of L-set and R-set of the rule are to be considered

# Information Cooking

- Let us consider that the states $s_{i-1}$ and $s_i$ are updated to $s'_{i-1}$ and $s'_i$ respectively. Now if $f(s_{i-1}, s_i, s_{i+1}) \neq f(s'_{i-1}, s'_i, s_{i+1})$, then it indicates that updates of itself and its left neighbor have an impact on cell $i$.
- Note that $s_{i-1}s_is_{i+1}$ and $s'_{i-1}s'_is_{i+1}$ are the two RMTs of an R-set
- To measure the amount of impact, we use the following equations.

$$\delta_r^{(+2)}(s) = \begin{cases} 1 & \text{if } f(r) \neq f(s),\ s \in \text{R-set(r)} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where, the amount of impact is defined as the function, $\delta_r^{(+2)}$ : R-set(r) $\rightarrow \{0, 1\}$.

# Information Cooking

one can now determine the number of times cell $i$ changes with the change of cell $i-1$ and cell $i$ as follows,

$$B_r^{(+2)} = \frac{1}{(d-1)^2} \sum_{s \in \text{R-set(r)}} \delta_r^{(+2)}(s) \quad (5)$$

Hence, the total possibility that cell $i$ will change with the change of cell $i-1$ and $i$ can be defined as,

$$\eta_c = \frac{1}{d^3} \sum_{r=0}^{d^3-1} B_r^{(+2)} \quad (6)$$

# Information Cooking

**Example**

*Let us illustrate the idea by ECA rule 102. To know the values of* $\delta_r^{(+2)}(s)$ *we need to know the next state values of the RMTs of R-set(r). Here, R-set (0)* $= \{6\}$*. The value of* $\delta_0^{(+2)}(3) = 1$ *because* $f(0) \neq f(3)$ *for this rule. Similarly, we get other values of* $\delta_r^{(+2)}(s)$*. Hence, we obtain the values of* $B_r^{(+2)}$*. Here,* $B_0^{(+2)} = B_1^{(+2)} = \cdots = B_7^{(+2)} = 1$*. Finally, we get the value of* $\eta_c$ *as 1. That is, the change of cell* $i-1$ *and cell* $i$ *affect the cell* $i$ *with probability 1.*

Using the same rationale, one can now find the measure, $\Lambda_c$, where $\Lambda_c$ denotes the information flow to the left when the cells $i+1$ and $i$ are updated. Here, we need L-set to find out $\Lambda_c$.

**Example**

*Consider the ECA rule 102. We can now calculate the information cooking due to the left (*$\Lambda_c$*). For the rule 102, the value of* $\Lambda_c = 0$*. This means the change of states of a cell and its right neighbor can not affect the cell.*

# Source of unpredictability

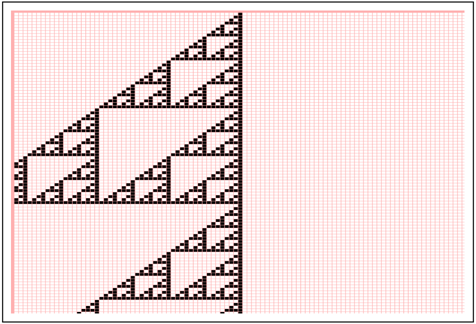
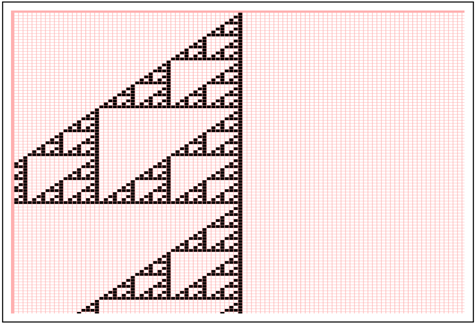


Figure: Space-time diagram of rule 102

- If all the cells affect their neighbors, that is, the values of $\Lambda_p$, $\eta_c$, $\Lambda_c$ and $\eta_p$ are non-zero for each $i \in \mathbb{Z}$, then any change in any cell would make the system unpredictable.

# Source of unpredictability



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- If all the cells affect their neighbors, that is, the values of $\Lambda_p$, $\eta_c$, $\Lambda_c$ and $\eta_p$ are non-zero for each $i \in \mathbb{Z}$, then any change in any cell would make the system unpredictable.

# The Communication class and chaos

- Any change in any cell should influence all the cells of the CA if the probability of affecting its neighbor is greater than 0, and there is no *blocking word* in the system. (Sometimes one cell may not communicate with another cell. In that case, the flow of information gets blocked.)
- Let us now define a relation *communicate* over set of cells of a CA
- We represent this relation by '$\rightsquigarrow$' where "$i \rightsquigarrow j$" means "$i$ communicates with $j$"
- Intuitively, this relation reflects the fact that a state change in cell $i$ has an influence on the cell $j$ in updating its state.

**Definition**

*A communication relation ($\rightsquigarrow$) is a relation from $\mathbb{Z}$ to $\mathbb{Z}$ where $i \rightsquigarrow j$ holds if and only if*
*1. $\eta_p > 0$ or $\eta_c > 0$ when $j > i$*
*2. $\Lambda_p > 0$ or $\Lambda_c > 0$ when $i > j$*
*3. both are true when $i = j$*

# The Communication class and chaos

- The relation $\rightsquigarrow$ is transitive by definition: if $i \rightsquigarrow j$ and $j \rightsquigarrow k$ then $i \rightsquigarrow k$
- Whether $\rightsquigarrow$ is reflexive or symmetric, however, depends on the given CA
- If $i \rightsquigarrow j$ but $j \not\rightsquigarrow i$, then it signifies that, any change in cell $i$ can influence the state of cell $j$ but any perturbation in cell $j$ has no effect on cell $i$.

To be a chaotic system, a tiny perturbation in initial condition should largely affect the future configurations of the system. Therefore, for each pair of $i, j \in \mathbb{Z}$, "if $i \rightsquigarrow j$, then $j \rightsquigarrow i$" has to hold in a chaotic CA. This implies, the relation '$\rightsquigarrow$' has to be an equivalence relation, and all the cells have to form a single equivalence class. Let us call this class as *communication class*

# The Communication class and chaos

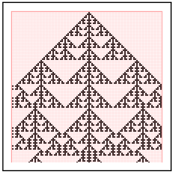

Figure: ECA rule 150 that forms a communication class

The figure shows the space-time diagram of ECA rule 150. Here each cell communicates with other cell and thus all the cells are in communication relation among themselves. Thus this rule forms a communication class.

# The Communication class and chaos

To be a chaotic CA, all of its cells have to communicate among themselves. Thus communication can not be blocked by a blocking word. Hence, one of the properties of a chaotic CA can be given as follows.

**Property**

*All the cells of a chaotic CA form a single communication class.*

A CA, which is a dynamical system under its global transition function $F$, is topologically transitive if for any two non-empty open subsets $U$ and $V$ of $S^{\mathbb{Z}}$, there exists a natural number $t$ such that $F^t(U) \cap V \neq \emptyset$. If a CA is not transitive, it can not be chaotic. And, if the CA does not form a single communication class, it can not be in general transitive.

**Proposition**

*If a CA is transitive and $\rightsquigarrow$ is an equivalence relation, then all of its cells form a single communication class.*

# Parametrization

In order to determine the possibility of creating disturbances to the left and right direction of the system, we have considered two parameters - **L** and **R** respectively.

$$\mathbf{L} \equiv (\Lambda_p, \max(\eta_p, \eta_c)) \tag{7}$$

$$\mathbf{R} \equiv (\eta_p, \max(\Lambda_p, \Lambda_c)) \tag{8}$$

where, $\mathbf{L} \geq \mathbf{R}$ holds if $\Lambda_p > \eta_p$. If $\Lambda_p = \eta_p$, then $\mathbf{L} \geq \mathbf{R}$ holds if $\max(\eta_p, \eta_c) \geq \max(\Lambda_p, \Lambda_c)$.
Thus if the disturbance has been created in either direction, the dynamics of the system will become chaotic. Therefore, the probability that the system will become chaotic can be defined by,

$$p = \max(\mathbf{L}, \mathbf{R}) \tag{9}$$

# Parametrization

The behavior becomes simple to chaotic as we increase the parameter value $p$. If the value of $a$ is 0, then there is no information flow and the system is simple. However, once we obtain the value of $a$, then we will consider the value of $b$. If the value of $a$ is high (i.e. tends to 1) and the value of $b$ is also high, the CA can be said as a chaotic CA.

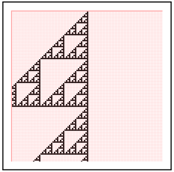

Figure: ECA rule 102 where $p = (1, 1)$

## Example

*$\Lambda_p = \eta_c = 1$ and $\eta_p = \Lambda_c = 0$ for ECA 102. So, we obtain $\mathbf{L} \equiv (1, 1)$ and $\mathbf{R} \equiv (0, 1)$. Here, $\mathbf{L} \geq \mathbf{R}$. Thus the probability of disturbance in the system is, $p = \mathbf{L}$. The value of $p$ is (1, 1). Therefore, the CA is a chaotic CA.*

# Parametrization

## Example

*Consider ECA rule 15. Here, $\Lambda_p = \Lambda_c = 0$ and $\eta_p = \eta_c = 1$. Hence, $\mathbf{L} \equiv (0, 1)$ and $\mathbf{R} \equiv (1, 0)$ and $p = (1, 0)$. As the second argument of $p$ is 0, the disturbance can not be created within the system and hence the CA is a simple CA. In fact, the cells do not form a single communication class.*

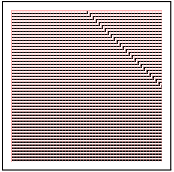

Figure: ECA rule 15 where $p = (1, 0)$

From experimental results it is seen that, if the value of $a \geq 0.75$ and the value of $b \geq 0.5$, the behavior becomes chaotic for most of the cases. Hence, if two parameter values for two CAs are given such that $p_1 = (1, 0.25)$ and $p_2 = (0.75, 0.75)$, then even though the value of $a$ for $p_1$ is large, the CA with parameter $p_2$ is more chaotic than that of $p_1$.

## Conclusion

- The chaotic behaviors are determined according to what human brain perceives by observing the dynamics in space-time diagrams
- There are some existing parameter such as $\lambda$, $Z$-parameter, means field curve etc. to study the behavior of classical CAs.
- Our proposed parameter gives better result than any of the existing parameters

# Thank You